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# Robertson intelligent states

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**Abstract.** The diagonalization of the uncertainty matrix and the minimization of Robertson inequality for $n$ observables are considered. It is proved that for even $n$ this relation is minimized in states which are eigenstates of $n/2$ independent complex linear combinations of the observables. In the case of canonical observables, this eigenvalue condition is also necessary. Such minimizing states are called Robertson intelligent states (RIS). The group-related coherent states (CS) with maximal symmetry (for semisimple Lie groups) are a particular case of RIS for the quadratures of Weyl generators. Explicit constructions of RIS are considered for operators of $su(1,1)$, $su(2)$, $h_N$ and $sp(N,R)$ algebras. Unlike the group-related CS, RIS can exhibit strong squeezing of group generators. Multimode squared amplitude squeezed states are naturally introduced as $sp(N,R)$ RIS. It is shown that the uncertainty matrices for quadratures of $q$-deformed boson operators $a_{q,j}$ ($q>0$) and of any $k$ power of $a_j = a_{1,j}$ are positive definite and can be diagonalized by symplectic linear transformations.

## 1. Introduction

Canonical coherent states (CS) [1, 2] in quantum optics and quantum mechanics can be defined in three equivalent ways: (1) as eigenstates of the non-Hermitian boson (photon) destruction operator $a$; (2) as orbits of the oscillator ground state $|0\rangle$ under the action of unitary displacement operator $D(\alpha)$; (3) as states which minimize the Heisenberg relation for canonical observables $q = (a+a^\dagger)/\sqrt{2}$ and $p = -\mathrm{i}(a-a^\dagger)/\sqrt{2}$ with equal uncertainties. Correspondingly, there are three ways of generalizing canonical CS [3]. The second one is the most general one considered, which consists of constructing orbits of a reference vectors $|\psi_0\rangle$ under the action of unitary operators of irreducible representations of a given Lie group [2, 3] ($D(\alpha)$ realize ray representation of Heisenberg–Weyl group $H_1$). These generalized CS are known (and should be called here) as group-related CS [2].

The main aim of this paper is to consider the third way of generalizing (the intelligence way) the case of $n$ observables and its relationship with the first two methods. The idea is to look for a generalized uncertainty relation for $n$ observables $X_\mu$, $\mu = 1, 2, \ldots, n$, which minimization would yield a continuous family of states such that in cases of $X_\mu$ being generators of a Lie group it would include the corresponding group-related CS.

It turns out that the required generic uncertainty relation (UR) for $n$ observables is that of Robertson [4], equation (1) (see also the review [5] on generalized uncertainty relations). Here we show that it is minimized in the eigenstates of $n/2$ (for even $n$) independent complex linear combinations of $X_\mu$ or (for any $n$) of at least one real combination. When $X_\mu$ are quadrature components of Weyl generators of a semisimple Lie group [6] these minimizing states contain as a subset the corresponding group-related CS with symmetry [2, 3]. Thus, it is the Robertson relation that naturally connects the above three ways of generalization of CS on the level of $n$ observables. In the case of the $N$ mode electromagnetic field we get that Robertson UR (RUR) is minimized if and only if the state is an eigenstate of $N$ new destruction operators $a'_j = u_{jk}a_k + v_{jk}a_k^\dagger$. For brevity, states which minimize some uncertainty relation should be called here *intelligent states* (IS) (the term IS was introduced in [7] in the example of spin states which minimize Heisenberg UR). The term *correlated* [8] is reserved for states with nonvanishing covariances (correlations).

The first step in the first method of CS generalization was made in [9–11] where eigenstates of complex combinations of $a_j$ and $a_j^\dagger$, $j = 1, 2, \ldots, N$, were constructed and discussed ($N = 2$ in [9], $N = 1$ in [10], any $N$ in [11]). Later [8], it was shown that eigenstates of $ua+va^\dagger$ minimize the Schrödinger UR (SUR) [12] for $q$ and $p$ (equation (3)), the minimizing states being called *correlated* CS. Those CS are in fact the same [13] as canonical squeezed states in quantum optics [14]. In [15] it was proved that SUR for any two observables $X$ and $Y$ is minimized in eigenstates of their complex combination $\lambda X + \mathrm{i}Y$ (equivalently of $uA + vA^\dagger$, $A = (X + \mathrm{i}Y)/\sqrt{2}$, $\lambda$ and $u, v$ are complex numbers). Eigenstates of $uA + vA^\dagger$ can exhibit strong squeezing in $X$ and $Y$. Schrödinger IS (SIS) for the generators $K_1, K_2$ of $SU(1,1)$ were constructed in [15] and shown to combine the Barut–Girardello CS [16] and $SU(1,1)$ group-related CS with symmetry [3]. The full sets of even and odd SIS for quadratures of squared boson destruction operator $a^2$ were constructed in the second paper of [19] (see also [31, 30]). Eigenstates of $uA + vA^\dagger$ with real $u$ and $v$ are noncorrelated SIS, that is Heisenberg IS, and the cases of $X, Y$ being quadratures of $aa$ or of the product $ab$ of two annihilation operators were considered in [17].

Another purpose of this paper is to consider the diagonalization problem of the uncertainty matrix, denoted here by $\sigma$. This matrix is of direct physical significance since its elements are dispersions (variances) and correlations (covariances) of observables. $\sigma$ is also important in quantum-state geometry [18]. In the case of canonical operators $q_j$, $p_k$ diagonalization of $\sigma$ was considered in [19].

The paper is organized as follows. In section 2 we briefly review Robertson relations for the uncertainty matrix $\sigma$ for $n$ observables $X_\mu$. In section 3 we consider the diagonalization of $\sigma$ by means of linear transformations of $X_\mu$. We note that in any state $\sigma$ can be diagonalized by means of the orthogonal transformation. From this it follows that the spin-component correlations can be eliminated by coordinate rotation. When the uncertainty matrix is positive definite (as is the case of $2N$ quadratures of $k$ power of boson/photon annihilation operators $a_j$ and the case of quadratures of $q$-deformed boson operators $a_{j,q}$ for $q > 0$) it can also be diagonalized by means of symplectic transformation. A new family of trace class UR (15) is established for positive definite dispersion matrices.

In section 4 we study the minimization of $n$-dimensional RUR. In section 5 explicit examples of RIS are considered, the $su(1,1)$ and $su(2)$ RIS being discussed in greater detail. RIS for generators of $SU(1,1)$ in the quadratic bosonic representation can exhibit linear and quadratic amplitude squeezing (even simultaneously—joint squeezing of two noncommuting observables).

## 2. Robertson uncertainty inequalities

For $n$ observables (Hermitian operators) $X_\mu$ Robertson [4] (see also [5]) established the following two uncertainty relations for the dispersion matrix $\sigma$,

$$\det\sigma \geqslant \det C \tag{1}$$

$$\sigma_{11}\sigma_{22}\ldots\sigma_{nn} \geqslant \det\sigma \tag{2}$$

where $\sigma_{\mu\nu} = \langle X_\mu X_\nu + X_\nu X\mu \rangle/2 - \langle X_\mu \rangle \langle X_\nu \rangle$ and $C$ is the antisymmetric matrix of mean commutators, $C_{\mu\nu} = -(\mathrm{i}/2)\langle [X_\mu, X_\nu] \rangle$. Here $\langle X \rangle$ is the mean value of $X$ in quantum state $\rho$, which is generally a mixed state. For $n = 2$ inequality (1) coincides with SUR ($X_1 = X$, $X_2 = Y$)

$$\Delta^2 X \Delta^2 Y - \sigma^2 XY \geqslant \tfrac{1}{4} |\langle [X, Y] \rangle|^2 \tag{3}$$

which in turn is reduced to the Heisenberg UR for $X$ and $Y$ when the covariance $\sigma_{XY} = \langle XY + YX \rangle/2 - \langle X \rangle \langle Y \rangle$ is vanishing ($\Delta^2 X \equiv \sigma_{XX}$, $\Delta^2 Y \equiv \sigma_{YY}$).

Combining (1) and (2) one obtains

$$\sigma_{11}\sigma_{22} \ldots \sigma_{nn} \geqslant \det C \tag{4}$$

which can be treated as a direct extension of Heisenberg UR to the case of $n$ operators.

The uncertainty matrix (dispersion or correlation matrix) $\sigma = \sigma(\boldsymbol{X}, \rho)$, where $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$, is symmetric by construction. From Robertson inequalities (1) and (2) one can deduce that its determinant is always nonnegative. Indeed, the matrix of mean commutators is antisymmetric and the determinant of the antisymmetric matrix is nonnegative [20]. Thereby in any state $\rho$ we have $\det C \geqslant 0$. $\det C$ vanishes identically if the number of operators $n$ is odd.

Diagonal elements of $\sigma$ are the variances of $X_\mu$. The problem of reducing (squeezing) of variances of quantum observables is important in physics (in quantum optics [14]) of precise measurements and telecommunications. The nondiagonal elements are the covariances of $X_\mu$ and $X_\nu$ and describe $X_\mu$–$X_\nu$ correlations. The uncertainty matrix in pure state $|\psi_0\rangle$ can be used as a metric tensor in the manifold of generalized Glauber CS $D(\alpha)|\psi_0\rangle$ [18]. In view of these dynamical and geometrical properties of $\sigma$ it is desirable to study the problem of its diagonalization (which is equivalent to the problem of minimizing the second Robertson relation (2)). Diagonalization of $\sigma$ in the case of canonical observables $p_j = X_j$, $q_j = X_{N+j}$, $j = 1, 2, \ldots, N$, was recently considered in [19]: in any state it can be diagonalized by means of linear canonical transformations. In section 3 we consider this problem in more general cases. The minimization of (1) for two observables $X_1$ and $X_2$ (i.e. of SUR (3)) has been shown [15] to occur in the eigenstates of their complex (in particular real) linear combinations only. In section 4 we extend this result to arbitrary $n$.

## 3. Diagonalization of uncertainty matrix of $n$ observables

In this section we consider the diagonalization of the uncertainty matrix $\sigma(\boldsymbol{X}, \rho)$ by means of linear transformations of $n$ operators $X_\mu$ (summation over repeated indices),

$$X_\mu \to X'_\mu = \lambda_{\mu\nu} X_\nu \tag{5}$$

where $\lambda_{\mu\nu}$ are real numbers (in order for $X'_\mu$ to be Hermitian operators again).

We first note the transformation property of $\sigma$ under transformation (5). Defining the new matrix $\sigma'$ as $\sigma' = \sigma(\boldsymbol{X}', \rho)$ we easily obtain

$$\sigma' = \Lambda \sigma \Lambda^T \tag{6}$$

where we introduce the $n$ vector $\boldsymbol{X}' = (X'_1, \ldots, X'_n)$ and $n \times n$ matrix $\Lambda = \{\lambda_{\mu\nu}\}$, its transpose being denoted as $\Lambda^T$. Thus, the two dispersion matrices are congruent via the transformation matrix $\Lambda$. We suppose that transformation (5) is invertable and set $\det \Lambda = 1$. In matrix form equation (5) is rewritten as $\boldsymbol{X}' = \Lambda \boldsymbol{X}$.

We note several general properties of $\sigma$, some of which are immediate consequences of its symmetricity and the transformation law (6). First, we note the invariant quantities: (a)

$\det\sigma = \det\sigma'$ for any $\Lambda \in SL(n, C)$; (b) $\mathrm{Tr}\,\sigma^k = \mathrm{Tr}\,\sigma'^k$, $k = 1, 2, \ldots$, for orthogonal $\Lambda$; (c) $\mathrm{Tr}(\sigma J)^k = \mathrm{Tr}(\sigma' J)^k$ for symplectic transformations ($n = 2N$),

$$\Lambda J \Lambda^T = J \qquad J = \begin{pmatrix} 0 & 1_N \\ -1_N & 0 \end{pmatrix}. \tag{7}$$

The last two invariants are particular cases of quite general relations, $\mathrm{Tr}(\sigma' g)^k = \mathrm{Tr}(\sigma g)^k$ which hold for $\Lambda$ satisfying $\Lambda^T g \Lambda = g$ with any fixed matrix $g$ (in the above $g = 1$ and $g = J$).

Next we note that $\sigma$ (being symmetric) can always be diagonalized by means of orthogonal $\Lambda$ ($\Lambda\Lambda^T = 1$) [20] in any state, i.e. $\sigma'$ is diagonal for some orthogonal $\Lambda$. In the case of spin (or angular momentum) operators, we obtain from this property that spin-component correlations can be considered as pure coordinate effects. Another general property of $\sigma$ is its nonnegativity, $\sigma \geqslant 0$. To prove this last property we diagonalize $\sigma$ by means of the orthogonal matrix $\Lambda$. The new operators $X'_\mu$, equation (5), are again Hermitian and therefore all the diagonal elements of the matrix $\sigma'$ are nonnegative. Therefore $\sigma \geqslant 0$ in any state $\rho$.

Further properties of the uncertainty matrix can be established when the set of operators $X_\mu$ possess some additional properties. For example, if $\sigma$ is positive definite, $\sigma > 0$, then it can be diagonalized by means of *symplectic* $\Lambda$ [21]. Therefore it is important to know when the uncertainty matrix is strictly positive. The value of $\det C \geqslant 0$ turned out to play an important role. Note that $\det\sigma > 0$ stems from $\sigma > 0$ and $\det\sigma = 0$ means that $\sigma$ is not strictly positive.

*Proposition 1.* $\det\sigma(\boldsymbol{X}, \rho) = 0$ in pure states $\rho = |\psi\rangle\langle\psi|$ iff $|\psi\rangle$ is an eigenstate of a real combination $\lambda_\nu X_\nu$ of $X_\nu$.

*Proof.* (a) Necessity. Let $\det\sigma(\boldsymbol{X}, \rho) = 0$. Then orthogonal $\Lambda$ exists such that $\sigma'$ is diagonal. We have $0 = \det\sigma = \det\sigma' = \sigma'_{11}\sigma'_{22}\ldots\sigma'_{nn}$, wherefrom at least for one $\nu$ one has $\sigma'_{\nu\nu} = 0$. The latter is possible in pure states $\rho = |\psi\rangle\langle\psi|$ if and only if $X'_\nu|\psi\rangle = x'_\nu|\psi\rangle$. (b) Sufficiency. Let $(\boldsymbol{\lambda X})|\psi\rangle = x'|\psi\rangle$, $\boldsymbol{\lambda X} \equiv \lambda_\nu X_\nu$. Then we can always construct the nondegenerate matrix $\Lambda$ with first row $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and consider the uncertainty matrix $\sigma' = \sigma(\Lambda\boldsymbol{X}; \psi)$. This $\sigma'$ is with a vanishing determinant since the first column of it is zero (as a consequence of $(\boldsymbol{\lambda X})|\psi\rangle = x'|\psi\rangle$). But $0 = \det\sigma' = (\det\Lambda)^2\det\sigma$, therefore $\det\sigma = 0$.

In view of this proposition and equation (1) one has the following.

*Corollary 1.* If $\det C(\boldsymbol{X}, \psi) > 0$ then $|\psi\rangle$ cannot be a normalizable eigenstate of any real combination $\lambda_\nu X_\nu$.

If $\det C(\boldsymbol{X}; \psi) > 0$ in any state then neither $X_\mu$ nor any real combination $\lambda_\nu X_\nu$ can be diagonalized in Hilbert space of states $\mathcal{H}$, that is the spectrum of $X_\mu$ and $\lambda_\nu X_\nu$ are continuous. Here is a class of $2N$ operators for which $\det C > 0$ and therefore $\sigma$ is positive in any state.

*Proposition 2.* If $X_\mu$, $\mu = 1, 2, \ldots, 2N$ obey the commutation relations

$$[X_j, X_{N+k}] = \delta_{jk}[X_j, X_{N+j}] \qquad [X_j, X_k] = 0 = [X_{N+j}, X_{N+k}] \tag{8}$$

where $-\mathrm{i}[X_j, X_{N+j}]$ are positive definite operators, then $\det C(\boldsymbol{X}; \rho) > 0$ and the uncertainty matrix $\sigma(\boldsymbol{X}; \rho)$ is positive definite.

*Proof.* By direct calculations we obtain

$$\det C = (\tfrac{1}{2})^{2N} \langle(-\mathrm{i})[X_1, Y_1]\rangle^2 \langle(-\mathrm{i})[X_2, Y_2]\rangle^2 \ldots \langle(-\mathrm{i})[X_N, Y_N]\rangle^2 \tag{9}$$

where $Y_j \equiv X_{N+j}$, $j = 1, 2, \ldots, N$. Since every factor in (9) is positive one has $\det C > 0$. From corollary 1 and the diagonalization of $\sigma$ by orthogonal $\Lambda$ we derive that $\det C > 0$ is a sufficient condition for $\sigma$ to be positive definite. □

We can point out a family of the boson system (e.g. $N$ mode electromagnetic field) observables which obey the commutation relations (8). Those are the quadrature components of power $k$ of photon (boson) destruction operators $a_j$, defined here as

$$X_j^{(k)} = \frac{1}{\sqrt{2k}}(a_j^k + a_j^{k\dagger}) \qquad X_{N+j}^{(k)} = \frac{-\mathrm{i}}{\sqrt{2k}}(a_j^k - a_j^{k\dagger}) \equiv Y_j. \tag{10}$$

Relations (8) and the positivity of $-\mathrm{i}[X_j^{(k)}, Y_j^{(k)}] = (1/k)[a^k, a^{k\dagger}]$ can be checked by direct calculations. As a result, the quadrature components of $a^k$ are continuous observables, their uncertainty matrix is positive definite and can be diagonalized by means of symplectic $\Lambda$. For $k = 1$, operators (10) are the canonical pairs $q_j$, $p_j$, therefore their uncertainty matrix can be diagonalized by means of linear canonical transformations, corresponding to symplectic $\Lambda$. The procedure for diagonalizing a positive definite matrix by means of symplectic $\Lambda$ is described in [21] and in the first paper of [19]. Canonical transformations with time-dependent $\Lambda(t)$ can be used to diagonalize any quadratic Hamiltonian. For an oscillator with varying mass and/or frequency this was done by Seleznyova [22].

Positive definite uncertainty matrices also exist in $q$-deformed boson systems. A $q$-deformed oscillator was introduced in [23]. The deformed lowering and raising operators $a_q$ and $a_q^\dagger$ obey the commutation relation

$$[a_q, a_q^\dagger] = [N_q + 1] - [N_q] \qquad [N] \equiv \frac{q^N - q^{-N}}{q - q^{-1}} \tag{11}$$

where $N_q$ is a number operator whose eigenstates are $|n\rangle_q = ([n]!)^{-1/2} a_q^{\dagger n} |0\rangle_q$: $N_q|n\rangle_q = n|n\rangle_q$, $a_q|0\rangle_q = 0$, $[n]! = [n][n-1]\ldots[1]$. At $q = 1$, $a_q, a_q^\dagger$ coincide with ordinary boson operators $a, a^\dagger$. Now we note that the commutator $[a_q, a_q^\dagger]$ is positive definite for $q > 0$ as one can easily verify by using (11). From the commutation relations for $n$ $q$-deformed oscillators [24]

$$\begin{aligned} &[a_{q,j}, a_{q,k}] = 0 \qquad [a_{q,j}, a_{q,k}^\dagger] = \delta_{jk}[a_{q,j}, a_{q,j}^\dagger] \\ &[N_{q,j}, a_{q,k}] = -\delta_{jk} a_{q,k} \qquad [N_{q,j}, a_{q,k}^\dagger] = \delta_{jk} a_{q,j}^\dagger \end{aligned} \tag{12}$$

it follows that the set of quadrature components of $a_{q,j}$ obey the requirements of proposition 2 for $q > 0$. Therefore the uncertainty matrix $\sigma(\boldsymbol{X}_q; \rho)$ is positive definite in any state for $q > 0$.

For the positive definite uncertainty matrix of $2N$ observables satisfying (8) one can establish a set of new uncertainty relations. For this purpose consider the invariant quantities $\mathrm{Tr}(\mathrm{i}\sigma J)^{2k}$, $k = 1, 2, \ldots$. Let $\sigma'$ be a diagonal matrix which is symplectically congruent to $\sigma$. Then we have

$$\mathrm{Tr}(\mathrm{i}\sigma J)^{2k} = \mathrm{Tr}(\mathrm{i}\sigma' J)^{2k} = 2\sum_j^N [\sigma'_{j,j}\sigma'_{N+j,N+j}]^k. \tag{13}$$

In view of $\sigma > 0$, every term $\sigma'_{j,j}\sigma'_{N+j,N+j}$ in (13) is nonvanishing and positive. We can apply the Heisenberg relation for $\sigma'_{j,j}\sigma'_{N+j,N+j}$ and write the set of inequalities

$$\mathrm{Tr}(\mathrm{i}\sigma J)^{2k} \geqslant 2\sum_{j}^{N} |\tfrac{1}{2}\langle[X'_j, X'_{N+j}]\rangle|^k. \tag{14}$$

In the above $X'_\mu = \Lambda_{\mu\nu}(\rho)X_\nu$ and $\Lambda(\rho)$ is the diagonalizing symplectic matrix for the state $\rho$. For every state we can in principle find the minimal value $c_0^2(\rho)$ of the $N$ terms $\left|\langle[X'_j, X'_{N+j}]\rangle\right|$ and therefore rewrite (14) in a more compact form

$$\mathrm{Tr}(\mathrm{i}\sigma J)^{2k} \geqslant \frac{N}{2^{2k-1}}c_0^{2k} \qquad k = 1, 2, \ldots. \tag{15}$$

In the particular case of canonical variables $X_j = p_j$, $X_{N+j} = q_j$ in any state the products $\sigma'_{j,j}\sigma'_{N+j,N+j}$ are greater than or equal to $\frac{1}{4}$ (this is the value of $\sigma'_{j,j}\sigma'_{N+j,N+j}$ in Glauber CS for mode $j$, $\hbar = 1$), that is $c_0^2 \geqslant 1$. Thus, for canonical variables the above UR read ($\boldsymbol{Q} = (p_1, \ldots, p_N, q_1, \ldots, q_N)$)

$$\mathrm{Tr}[\mathrm{i}\sigma(\boldsymbol{Q}, \rho)J]^{2k} \geqslant \frac{N}{2^{2k-1}}. \tag{16}$$

The latter inequalities for the case of $\sigma(\boldsymbol{Q}, \rho)$ (apart from the factor i) were recently obtained by Sudarshan *et al* [19]. For $N = 1$ and $k = 1$ inequality (16) recovers the Schrödinger relation (3).

The above-considered diagonalization of the uncertainty matrix of $n$ Hermitian operators by means of transformations of operators $X_\mu \to X'_\mu$ should be referred to here as first-kind diagonalizations. The state $\rho$ here is kept the same. This diagonalization is always possible as we have shown. But it is also of interest to know when $\sigma$ can be diagonalized by state transformation, keeping observables the same. That is for given $X_\mu$ and state $\rho$ find the new state $\rho'$ so that the new matrix $\sigma'' \equiv \sigma(\boldsymbol{X}, \rho')$ is diagonal. We shall call this second-kind diagonalization. Evidently, both diagonalizations coincide (i.e. $\sigma' = \sigma''$) when transformation (5) is generated by some unitary operator $U(\Lambda)$,

$$X'_\mu = \lambda_{\mu\nu}X_\nu = U^\dagger(\Lambda)X_\mu U(\Lambda). \tag{17}$$

Such is the case for example of uncertainty matrix $\sigma(\boldsymbol{Q}, \rho)$ of canonical operators $p_j \equiv Q_j$ and $q_j \equiv Q_{N+j}$ when the diagonalizing $\Lambda$ is symplectic. Then $U(\Lambda)$ is a representation of the group $Sp(N, R)$ [6] (more precisely of $Mp(N, R) = \overline{Sp(N, R)}$) and thus any pure or mixed canonical correlated state is unitary equivalent to the noncorrelated state. In the case of $N = 1$ we have an extra diagonalizing property: in view of the fact that the squared boson operators $a^2$, $a^{\dagger 2}$, $a^\dagger a$ close the $su(1, 1)$ algebra, equation (52) ($su(1, 1) \sim sp(1, R)$) we get that in the one-mode field case the quadratic amplitude dispersion matrix is also diagonalizable by unitary $Mp(1, R)$ state transformation. Property (17) also occurs in the cases when $X_\mu$ close orthogonal algebra $so(n, R)$. Then the diagonalizing orthogonal transformation (5) is generated by unitary $U(\Lambda) \in SO(n, R)$. In the example of $so(3, R) \sim su(2)$ this means (recall that if $J_k$ are spin operators, $[J_k, J_j] = \mathrm{i}\hbar\epsilon_{kjl}J_l$, and $\Lambda$ is orthogonal then $[J'_k, J'_j] = \mathrm{i}\hbar\epsilon_{kjl}J'_l$) that spin-component correlations (covariances) in any state can always be eliminated by means of coordinate rotation (first-kind diagonalization) and by state transformation with unitary operator $U(\Lambda)$ (second-kind diagonalization). In other words spin-component correlation is a pure coordinate effect and any spin-correlated state is unitary equivalent to a noncorrelated one.

## 4. Minimization of Robertson uncertainty inequality $\det\sigma \geqslant \det C$

One general sufficient condition for the minimization of Robertson inequality (1) for arbitrary observables $X_\mu$, $\mu = 1, 2, \ldots, n$, follows from proposition 1: the equality in (1) holds in the eigenstates of at least one of $X_\mu$ since in such a case both matrices $\sigma$ and $C$ have at least one vanishing column and then $\det\sigma = \det C = 0$. In view of the fact that $\sigma$ can always be digonalized by means of orthogonal $\Lambda$ (second immediate property in section 2) the minimization of both Robertson relations for any $n$ also occurs in the eigenstates of some of $X'_\mu = \lambda_{\mu\nu} X_\nu$.

In the case of odd $n$ the above sufficient condition for the minimization of (1) is also a necessary one. Inequality (1) is minimized in a state $|\psi\rangle$ if and only if $|\psi\rangle$ is the eigenstate of a real combination $\lambda_\nu X_\nu$ of observables $X_\nu$. The proof follows from proposition 1 and the property of the determinant of antisymmetric matrices of an odd dimension: for odd $n$ $\det C$ of antisymmetric matrix $C$ is vanishing identically in any state.

$\det C$ can only be greater than 0 for even $n$. For an even number of operators $X_\mu$ we establish the following sufficient condition.

*Proposition 3.* The equality in the RUR (1) for $2N$ Hermitian operators $X_\mu$ holds in the eigenstates $|\psi\rangle$ of $N$ independent complex linear combinations of $X_\mu$.

*Proof.* Let $X'_\mu = \lambda_{\mu\nu} X_\nu \equiv X'_\mu(\Lambda)$ be some linear transformation which preserves the hermiticity, i.e. $\lambda_{\mu\nu}$ are real parameters. We introduce $N$ non-Hermitian operators $A_j = X_j + \mathrm{i}X_{N+j}$ and construct $N$ independent complex combinations of all $X_\nu$ in the form,

$$A'_j = X'_j + \mathrm{i}X'_{N+j} = u_{jk} A_k + v_{jk} A_k^\dagger \tag{18}$$

where $u_{jk}$ and $v_{jk}$ are new complex parameters which are simply expressed in terms of $\lambda_{\mu\nu}$ ($j, k = 1, 2, \ldots, N$). Now let $|\psi\rangle$ be eigenstate of all $A'_j$,

$$A'_j|\psi\rangle = z_j|\psi\rangle \qquad j = 1, 2, \ldots, N \tag{19}$$

$z_j$ being the eigenvalue. It is natural to denote the solutions of (19) as $|\boldsymbol{z}, u, v\rangle$ or equivalently as $|\boldsymbol{z}, \Lambda\rangle$, where $u, v$ are $N \times N$ matrices and $\Lambda$ is $2N \times 2N$.

The scheme of the proof is to express both matrices $\sigma(\boldsymbol{X}, \psi)$ and $C(\boldsymbol{X}, \psi)$ in terms of matrices $\sigma(\boldsymbol{B}', \psi)$ and $C(\boldsymbol{B}', \psi)$ and to compare their determinants. Here $\boldsymbol{B} = (A_1, A_2, \ldots, A_N, A_1^\dagger, A_2^\dagger, \ldots, A_N^\dagger) \equiv (\boldsymbol{A}, \boldsymbol{A}^\dagger)$ and $\boldsymbol{B}' = (\boldsymbol{A}', \boldsymbol{A}'^\dagger)$. First we relate $\boldsymbol{X}$ to $\boldsymbol{B}$,

$$\boldsymbol{X} = b\boldsymbol{B} \qquad b = \frac{1}{2}\begin{pmatrix} 1_N & 1_N \\ \mathrm{i}1_N & -\mathrm{i}1_N \end{pmatrix} \tag{20}$$

where $1_N$ is $N \times N$ unit matrix. We introduce $2N \times 2N$ transformation matrix $V$, which relates $\boldsymbol{B}$ and $\boldsymbol{B}'$,

$$\boldsymbol{B}' = V\boldsymbol{B} \qquad V = \begin{pmatrix} u & v \\ v^* & u^* \end{pmatrix} \tag{21}$$

where $u$ and $v$ are $N \times N$ matrices of transformation (18). We consider the new operators $A'_j$ independent (as well as the old ones $A_j$), therefore matrix $V$ is supposed to be invertable, that is $\det V \neq 0$. Using the above two linear transformations and the definition of $\sigma$ we obtain

$$\sigma(\boldsymbol{X}, \psi) = bV^{-1}\sigma(\boldsymbol{A}', \psi)(V^{-1})^T b^T \tag{22}$$

and similarly

$$C(\boldsymbol{X}, \psi) = bV^{-1}C(\boldsymbol{B}', \psi)(V^{-1})^T b^T. \tag{23}$$

Next, using the eigenvalue equations (19) we can prove the equality

$$\det \sigma(\boldsymbol{B}', \psi) = \det C(\boldsymbol{B}', \psi) \tag{24}$$

which in view of (22) and (23) (and the nondegeneracy of $b$ and $V$, $\det b = (-\mathrm{i}/2)^N$) leads to the desired equality in the RUR (1),

$$\det \sigma(\boldsymbol{X}, \psi) = \det C(\boldsymbol{X}, \psi). \tag{25}$$

The proof of auxiliary equality (24) can be carried out by direct calculations: one has

$$\begin{aligned}
&\sigma_{jk}(\boldsymbol{B}', \psi) = 0 = C_{jk}(\boldsymbol{B}', \psi)\\
&\sigma_{N+j,N+k}(\boldsymbol{B}', \psi) = 0 = C_{N+j,N+k}(\boldsymbol{B}', \psi)\\
&\sigma_{j,N+k}(\boldsymbol{B}', \psi) = \mathrm{i}C_{j,N+k}(\boldsymbol{B}', \psi)\\
&\sigma_{N+j,k}(\boldsymbol{B}', \psi) = -\mathrm{i}C_{N+j,k}(\boldsymbol{B}', \psi)
\end{aligned} \tag{26}$$

which manifestly ensure (24). Thus, the states which satisfy equation (19) minimize inequality (1).

States which minimize RUR (1) for observables $(X_1, X_2, \ldots, X_n) \equiv \boldsymbol{X}$ should be called *Robertson intelligent states* for $\boldsymbol{X}$ (briefly $\boldsymbol{X}$-RIS). Equivalent terms could be Robertson minimum uncertainty states or Robertson correlated states, following for example [5, 8, 13]. However, we reserve the term correlated for states with nonvanishing correlations (covariances) only. In the case of even $n$, in view of (19) and (18), RIS should be denoted as $|\boldsymbol{z}, u, v\rangle$ or $|\boldsymbol{z}, \Lambda\rangle$. For $n = 2$ relation (1) coincides with the Schrödinger one, equation (3), and RIS are in fact SIS. For two observables condition (19) is necessary and sufficient [15] to obtain the equality in SUR.

Following the analogy to the known case of canonical observables $p_j$ and $q_j$ one can introduce the squeeze operator [3, 14, 33] for arbitrary observables (generalized squeeze operator) $S(u, v)$ as an operator which is a map from noncorrelated RIS with equal uncertainties for all pairs $X_j$ and $Y_j = X_{N+j}$ (those RIS minimize Heisenberg relation for $2N$ operators (4)) to correlated RIS (RIS with nonvanishing covariances and nonequal variances). Noncorrelated RIS with equal uncertainties for $X_j$ and $Y_j$ are obtained when $u_{jk} = \delta_{jk}$ and $v_{jk} = 0$ in $|\boldsymbol{z}, u, v\rangle$.

$$S(u, v) : |\boldsymbol{z}, u, v\rangle = S(u, v)|\boldsymbol{z}\rangle \tag{27}$$

where $|\boldsymbol{z}\rangle = |\boldsymbol{z},\ u = 1,\ v = 0\rangle$. $|\boldsymbol{z}\rangle$ are eigenstates of all $A_j$, $j = 1, 2, \ldots, N$. For two arbitrary observables the operator $S(u, v)$ was introduced in [15]. This definition is of importance for the generation of RIS $|\boldsymbol{z}, u, v\rangle$ from $|\boldsymbol{z}\rangle$ when the states $|\boldsymbol{z}\rangle$ are known and available. IS $|z\rangle$ with equal uncertainty for two observables $X$, $Y$ are constructed, in different notations, in a number of cases [1, 8, 16, 25–27]. It is interesting to note that for certain systems the squeeze operator $S(u, v)$ may exist as an isometric (not unitary) operator. Such is the case of $S(u, v)$ for the quadratures of squared boson annihilation operator $a^2$, considered in [31]. If $S$ is isometric then its generator $H$ (defined by $S = \exp(\mathrm{i}H)$) is symmetric (not Hermitian = selfadjoint) operator and can be considered as a generalized observable [28]. In such cases representing $S = \exp(\mathrm{i}tH)$ ($t$ being real parameter, the time) we see from (27) that RIS (for $n = 2$ in fact SIS) $|z, u, v\rangle$ can be generated from states with equal uncertainties $|z\rangle$ in a process of nonunitary evolution governed by symmetric Hamiltonian $H$. Symmetric but not selfadjoint is for example the particle momentum on a half line and the Hamiltonian of a particle with different mass parameters in $X$, $Y$ and $Z$ directions (moving in a crystal) [28].

Now a natural question of the existence of RIS arises. We have a positive answer to this question for a broad class of observables $X_\mu$: RIS exists for the operators of Hermitian representations of semisimple Lie algebras in Hilbert space $\mathcal{H}$ and for representations of solvable algebras $L$ in finite-dimensional $\mathcal{H}$. RIS may exist for infinite-dimensional representations of certain solvable algebras. The existence of RIS for any finite-dimensional representation of a solvable Lie algebra $L$ stems from the theorem [29] that any such representation possess at least one weight (i.e. a vector exists, which is the eigenvector of all elements of $L$).

## 5. Examples of RIS

### *5.1. RIS for semisimple Lie algebras*

First we note that for any Lie group $G$ the group related CS [2, 3] $|\psi(g)\rangle = U(g)|\psi_0\rangle$ with $|\psi_0\rangle$ being eigenvector of at least one generator $X_\mu$ (these are CS with symmetry) universally are RIS for the generators of $G$. Indeed, $U(g)|\psi_0\rangle$ is evidently an eigenstate of Hermitian operator $U(g)X_\mu U^\dagger(g)$ ($U(g)$ is a unitary representation of $G$). Then we can apply proposition 1 and get $\det\sigma(\boldsymbol{X};\psi(g)) = 0$. Here $\det C$ also vanishes identically with respect to $g \in G$, i.e. $\det\sigma(\boldsymbol{X};\psi(g)) = \det C(\boldsymbol{X};\psi(g)) = 0$. If $G$ is semisimple then Hermitian generators $H_l$ from Cartan subalgebra always have normalizable eigenvectors $|\psi_0\rangle$ [6]. Therefore CS $|\psi(g)\rangle$ with these $|\psi_0\rangle$ as reference vectors are RIS for all group generators (with the trivial minimization: $\det\sigma = \det C = 0$ identically with respect to $g \in G$).

We shall now prove that CS $|\psi(g)\rangle$ with maximal symmetry are RIS for the quadrature components of Weyl lowering operators $E_{-k}$ with the property $\det\sigma \geqslant 0$. The proof consists of an application of proposition 3. The number of quadrature components $X_k, Y_k$ of all $E_{-k}$ is even, denoted by $2n_w$, where $n_w$ is the number of Weyl operators $E_{-k}$: $E_{-k} = X_k - \mathrm{i}X_{n_w+k} \equiv X_k - \mathrm{i}Y_k$, $k = 1, 2, \ldots, n_w$. We shall prove that equation (19) (the sufficient condition for RIS) is satisfied by CS $|\psi(g)\rangle$. As operators $A_j$ we take here $E_{-k}$ and as $A'_j$ we have to take linear combinations of Weyl lowering and raising operators $u_{jk}E_{-k} + v_{jk}E_k$, $j, k = 1, 2, \ldots, n_w$ and then consider the eigenvalue equation

$$(u_{jk}E_{-k} + v_{jk}E_k)|\boldsymbol{z}, u, v\rangle = z_j|\boldsymbol{z}, u, v\rangle. \tag{28}$$

Consider the action of $u_{jk}E_{-k} + v_{jk}E_k$ on the state $|\psi(g)\rangle$. One has (summation over repeated indices, $E_k = E_{-k}^\dagger$, $H_l = H_l^\dagger$)

$$\begin{aligned}(u_{jk}E_{-k} + v_{jk}E_k)|\psi(g)\rangle &= (u_{jk}E_{-k} + v_{jk}E_k)U(g)|\psi_0\rangle \\ &= U(g)U^{-1}(g)(u_{jk}E_{-k} + v_{jk}E_k)U(g)|\psi_0\rangle \\ &= U(g)[(u_{jk}\tilde{u}_{ki} + v_{jk}\tilde{v}^*_{ki})E_{-i} + (u_{jk}\tilde{v}_{ki} + v_{jk}\tilde{u}^*_{ki})E_i \\ &\quad + (u_{jk}\tilde{w}_{kl} + v_{jk}\tilde{w}_{kl})H_l]|\psi_0\rangle.\end{aligned} \tag{29}$$

In the above we have applied the BCH formula to the transformations $U^{-1}E_kU$ ($k, j, i = 1, 2, \ldots, n_w$, $l, m = 1, 2, \ldots, n_c$, $n_c$ being the dimension of Cartan subalgebra)

$$U^{-1}(g)E_{-k}U(g) = \tilde{u}_{ki}E_{-i} + \tilde{v}_{ki}E_i + \tilde{w}_{kl}H_l. \tag{30}$$

Taking into account that $E_{-i}|\psi_0\rangle = 0$ and $H_l|\psi_0\rangle = h_l|\psi_0\rangle$ we see that $|\psi(g)\rangle$ should be an eigenstate of all $A'_j$ if the $n_w \times n_w$ matrices $u, v, \tilde{u}$ and $\tilde{v}$ satisfy the equation

$$u\tilde{v} + v\tilde{u}^* = 0. \tag{31}$$

In the last equation $\tilde{u} = \tilde{u}(g)$ and $\tilde{v} = \tilde{v}(g)$ should be treated as known for a given Lie group representation $U(g)$. Moreover, the matrix $\tilde{u}$ is nondegenerate. Therefore we can always solve equation (31), $v = -u\tilde{v}(g)\tilde{u}^{*-1}(g)$ and get $|\psi(g)\rangle$ as an eigenstate of $A'_j = u_{jk}E_{-k} + v_{jk}E_k$,

$$(u_{jk}E_{-k} + v_{jk}E_k)|\psi(g)\rangle = z_j|\psi(g)\rangle \tag{32}$$

with eigenvalues $z_j = (u_{jk}\tilde{w}_{kl} + v_{jk}\tilde{w}_{kl})h_l$. In view of (32), the group-related CS with maximal symmetry $|\psi(g)\rangle$ can be parametrized as RIS for $2n_w$ components of Weyl generators: $|\psi(g)\rangle = |\boldsymbol{z}, u, v\rangle$ where $u$ and $v$ are $n_w \times n_w$ matrices.

Thus, we have demonstrated that states from unitary (in particular unitary and irreducible) orbits of extremal weight vectors of semisimple Lie algebras are RIS for all basis operators $X_\mu$ and for the quadratures $X_k$, $Y_k = X_{n_w+k}$ of Weyl operators $E_{-k}$ as well. As far as we know, this intelligence property of the group-related CS has not been noted yet in the literature.

We stress that RIS for quadrature components of Weyl generators $E_{-k}$ are more general than the group-related CS with maximal symmetry: states $|\psi(g)\rangle$ are only a part of the set of solutions of eigenvalue equation (28), corresponding to constrain (31) on the parameters $u_{jk}$ and $v_{jk}$. In the example of $su(1,1)$ and $su(2)$ ($n_w = 1$, $n_c = 1$) this was analysed by explicit constructions of SIS $|z, u, v; k\rangle$ in [15].

It is worth noting that propositions 1 and 3 can be applied to any subset of the operators of a given Lie algebra $L$. Therefore it makes sense to consider the eigenvalue problem for general element of the complexified algebra $L^C$,

$$(\beta_\nu X_\nu)|\psi\rangle = z|\psi\rangle \tag{33}$$

where $X_\nu$ ($\nu = 1, \ldots, n$) are basis operators of $L$ and $\beta_\nu$ are complex parameters. Taking specific constrains on the complex parameters $\beta_\nu$ one can get various subset of RIS for less than $n$ algebra operators, in particular various $X_j$–$Y_k$ SIS. The property of group-related CS to be part of the set of eigenstates of complex linear combinations of all algebra operators was noted in [30, 31]. States that satisfy (33) could be called algebraic CS [31] or algebra eigenstates [30].

### *5.2. Explicit solutions for su(1, 1) and su(2) RIS*

First consider the $su(1,1)$ case. The basis elements of $su(1,1)$ are the three operators $K_\mu$, $\mu = 1, 2, 3$, which obey the relations

$$[K_1, K_2] = -\mathrm{i}K_3 \qquad [K_2, K_3] = \mathrm{i}K_1 \qquad [K_3, K_1] = \mathrm{i}K_2. \tag{34}$$

The Casimir operator is $C_2 = K_3^2 - K_2^2 - K_1^2 = k(k-1)$ and Weyl lowering and raising operators are $K_\mp = K_1 \mp \mathrm{i}K_2$. According to the previous discussion, RIS for all three algebra operators and for any pair $K_j$–$K_k$ are contained in the set of eigenstates of general element of the algebra. Therefore one has to consider the eigenvalue equation for the general element of $su^C(1,1)$,

$$(uK_- + vK_+ + wK_3)|z, u, v, w; k\rangle = z|z, u, v, w; k\rangle \tag{35}$$

where $u$, $v$, $w$ are complex parameters, simply related to $\beta_\nu$ introduced in (33). This equation can be solved [31, 30] using the Barut–Girardello CS representation (BG representation) [16] or the $SU(1,1)$ group-related CS representation [2, 3]. The solution can be carried out for $su(1,1)$ representations with Bargman index $k = \frac{1}{4}, \frac{3}{4}$ and for the discrete series $k = \frac{1}{2}, 1, \frac{3}{2}, \ldots$ (particular cases of $v = 0 = w$ and $w = 0$ were solved in [16, 15]. The

Barut–Girardello CS (BG CS) $|\eta; k\rangle$ are eigenstates of $K_-$: $K_-|\eta; k\rangle = \eta|\eta; k\rangle$. In this representation

$$K_+ = \eta \qquad K_- = 2k\frac{\mathrm{d}}{\mathrm{d}\eta} + \eta\frac{\mathrm{d}^2}{\mathrm{d}\eta^2} \qquad K_3 = k + \eta\frac{\mathrm{d}}{\mathrm{d}\eta} \tag{36}$$

and states $|\psi\rangle$ are represented by analytic functions $\Phi(\eta)$ which up to a certain common factor $f(|\eta|)$ are proportional to $\langle k; \eta^*|\psi\rangle$. Orthonormalized eigenstates $|m; k\rangle$ of $K_3$ are represented by monomials $\eta^m\,[\Gamma(k)/(m!\Gamma(m+k))]^{1/2}$. For $u \neq 0$ the required analytic solution of (35) is [31]

$$\Phi_z(\eta; u, v, w) = N(z, u, v, w)\exp(c\eta)M(a, b, c_1\eta) \tag{37}$$

where $N(z, u, v, w)$ is a normalization constant, $M(a, b, \eta)$ is the Kummer function (confluent hypergeometric function ${}_1F_1(a, b; \eta)$) [32], parameters $a, b, c$ and $c_1$ are

$$\begin{aligned} a &= k + \frac{z}{\sqrt{w^2 - 4uv}} \qquad b = 2k \\ c &= -\frac{1}{2u}\left(w + \sqrt{w^2 - 4uv}\right) \qquad c_1 = \frac{1}{u}\sqrt{w^2 - 4uv} \end{aligned} \tag{38}$$

and the normalizability conditions take the form

$$\frac{1}{2|u|}\left|w - \sqrt{w^2 - 4uv}\right| < 1 \qquad \text{or} \qquad \frac{1}{2|u|}\left|w + \sqrt{w^2 - 4uv}\right| < 1. \tag{39}$$

When inequalities (39) are broken down the functions $\Phi_z(\eta; u, v, w)$ are still solutions of equation (35) and could be considered as nonnormalizable eigenstates. In the case of $u = 0$ in equation (35), we have (in view of (36)) a first-order equation to solve [31]. It turned out that the solutions for this case could be obtained from $\Phi_z(\eta; u, v, w; k)$ taking appropriate limits in it. One can check that conditions (39) can be satisfied by real $w$ and $v = u^*$ when the operator $uK_- + vK_+ + wK_3$ becomes Hermitian. Then the algebraic states $|z, u, u^*, w\rangle$ ($w = w^*$) are RIS for the three observables $K_1$, $K_2$ and $K_3$. RIS for the nonsquare integrable representations corresponding to $k = \frac{1}{2}, \frac{3}{4}$ are considered in section 5.3.

Various known states are contained in the large family of $su(1,1)$ states $|z, u, v, w; k\rangle$ [31]. In particular, when $w = 0$ we get the SIS $|z, u, v; k\rangle$ for the noncompact generators $K_1$ and $K_2$, which in turn at $z = -k\sqrt{-uv}$ [15] recover the family of $SU(1,1)$ group CS $|\tau; k\rangle$ (the squeezed vacuum states [3]), $\tau = \sqrt{-v/u}$, $|\tau| < 1$. In view of the positivity of the commutator $\mathrm{i}[K_1, K_2] = [K_-, K_+]/2$ the uncertainty matrix $\sigma(K_1, K_2; \rho)$ is positive definite and therefore possesses the resulting properties, described in section 2. In IS $|z, u, v; k\rangle$ the matrix elements of $\sigma$ are

$$\sigma_{11} = \frac{1}{2}\frac{|u - v|^2}{|u|^2 - |v|^2}\langle K_3\rangle \qquad \sigma_{22} = \frac{1}{2}\frac{|u + v|^2}{|u|^2 - |v|^2}\langle K_3\rangle \qquad \sigma_{12} = \frac{\mathrm{Im}(u^*v)}{|u|^2 - |v|^2}\langle K_3\rangle \tag{40}$$

satisfying $\det\sigma = \det C = \langle K_3\rangle^2/4$. The $K_1$–$K_3$ and $K_2$–$K_3$ IS are obtained from $|z, u, v, w; k\rangle$ when $v = u$ and $v = -u$ respectively.

The case of $su(2)$, RIS (i.e. spin RIS) can be treated in a similar manner by using the representation of $SU(2)$ group related CS $|\zeta; j\rangle$ in which [3]

$$J_- = -\zeta^2\frac{\mathrm{d}}{\mathrm{d}\zeta} + 2j\zeta \qquad J_- = \frac{\mathrm{d}}{\mathrm{d}\zeta} \qquad J_3 = \zeta\frac{\mathrm{d}}{\mathrm{d}\zeta} - j. \tag{41}$$

Here $j = \frac{1}{2}, 1, \frac{3}{2}, \ldots$, $[J_-, J_+] = -2J_3$, $[J_3, J_\pm] = \pm J_\pm$ ($J_\pm = J_1 \pm \mathrm{i}J_2$) and $J^2 = J_1^2 + J_2^2 + J_3^2 = j(j+1)$. The required eigenvalue problem

$$(\beta_\nu J_\nu)|z, \boldsymbol{\beta}; j\rangle = z|z, \boldsymbol{\beta}; j\rangle \tag{42}$$

where $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3)$ ($\beta_\nu$ are complex parameters) was solved by Brif [30]. In the case of $b^2 \equiv \boldsymbol{\beta}\boldsymbol{\beta} \neq 0 \neq \beta_1 - \mathrm{i}\beta_2 \equiv \beta_-$ the solution is [30]

$$\Phi_z(\zeta; \boldsymbol{\beta}, j) = N_0 \left(\zeta - \frac{\beta_3 - b}{\beta_-}\right)^{j+z/b} \left(\zeta - \frac{\beta_3 + b}{\beta_-}\right)^{j-z/b} \tag{43}$$

with the normalizability condition $z = mb$, $m = -j, -j+1, \ldots, j-1, j$. As we expected, these $su(2)$ RIS contain the set of standard $SU(2)$ CS with maximal symmetry $|\zeta'; j\rangle$ and this occurs when $m = \pm j$ with $\zeta' = -\beta_-(\zeta_3 \mp b)^{-1}$ [30]. At $\beta_3 = 0$ the $su(2)$ RIS coincide with the Schrödinger $J_1$–$J_2$ IS considered in [15].

For the $su(2)$ observables $J_\nu$ (the spin components) it is important to note that the spin-component uncertainty matrix $\sigma(\boldsymbol{J}; \rho)$ in any state can be diagonalized by means of an orthogonal linear transformation of $J_\nu$. The latter can be induced by rotating coordinates in $\mathbb{R}_3$ since $su(2) \sim so(3)$. Therefore we deduce that spin-component correlations are of a pure coordinate nature—they can be eliminated in any state by rotations of the reference frame. Here one can also perform second-kind diagonalization of $\sigma$, keeping $J_\nu$ and transforming the state $\rho$ by an unitary operator $U(g)$ of $SU(2) \sim SO(3)$. Thus, correlated spin RIS are unitarily equivalent to noncorrelated spin RIS.

### *5.3. RIS of the multimode boson systems*

In this section we first consider $n = 2N$ canonical operators $p_j$ and $q_j$, $j = 1, 2, \ldots, N$, which are quadrature components of $N$ boson/photon destruction (creation) operators $a_j = (q_j + \mathrm{i}p_j)/\sqrt{2}$ ($a_j^\dagger = (q_j - \mathrm{i}p_j)/\sqrt{2}$): $[q_j, p_k] = \mathrm{i}\delta_{jk}$. Here for concreteness we put $X_\nu \equiv Q_\nu$, $Q_j = p_j$, $Q_{N+j} = q_j$ and $A_j = q_j + \mathrm{i}p_j = a_j\sqrt{2}$. The set of $Q_\mu$ and the unity operator close the Heisenberg algebra $h_N$, which is nilpotent (therefore nonsemisimple). So RIS for canonical observables $Q_\mu$ are $h_N$ RIS (to also be called the multimode amplitude RIS). According to proposition 3, eigenstates $|\boldsymbol{\alpha}, u, v\rangle \equiv |\boldsymbol{\alpha}, \Lambda\rangle$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_N)$, of $a_j'$,

$$a_j' = u_{jk}a_k + v_{jk}a_{jk}^\dagger = \tfrac{1}{2}[(\lambda_q)_{jk}q_k + (\lambda_p)_{jk}p_k] \tag{44}$$

with any $u$ and $v$ are RIS for $Q_\mu$,

$$a_j'|\boldsymbol{\alpha}; u, v\rangle = \alpha_j|\boldsymbol{\alpha}; u, v\rangle \qquad j = 1, 2, \ldots, N. \tag{45}$$

Here $u = (\lambda_q - \mathrm{i}\lambda_p)/2$, $v = (\lambda_q + \mathrm{i}\lambda_p)/2$ and $u, v, \lambda_q$ and $\lambda_p$ are $N \times N$ complex matrices. The $N \times N$ matrices $\lambda_q$ and $\lambda_p$ are related to the transformation matrix $\Lambda$ in (5) (which is now rewritten as $Q'_\mu = \lambda_{\mu\nu}Q_\nu$) as follows

$$\Lambda = \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{pmatrix} \qquad \lambda_p = \lambda_3 + \mathrm{i}\lambda_1 \qquad \lambda_q = \mathrm{i}\lambda_2 + \lambda_4. \tag{46}$$

If one imposes the symplectic conditions $\Lambda J \Lambda^T = J$ on $\Lambda$, the operators $a_j'$ become new annihilation operators, i.e. the linear transformation (5) becomes a canonical one. With this conditions $\boldsymbol{Q}$-RIS are unitary equivalent (with the methaplectic operator $U(\Lambda)$) to eigenstates of $a_j$, i.e. to the canonical multimode CS $|\boldsymbol{\alpha}\rangle$. In [11] states $|\boldsymbol{\alpha}, \Lambda(t)\rangle$ were constructed explicitly as a solution $|\boldsymbol{\alpha}; t\rangle$ of the time-dependent Schrödinger equation for general quadratic, possibly time-dependent, Hamiltonian $H = B_{\mu\nu}(t)Q_\mu Q_\nu$ (plus linear terms as well). In terms of parameter matrices $\lambda_q$ and $\lambda_p$ these canonical RIS $|\boldsymbol{\alpha}, \Lambda\rangle = |\boldsymbol{\alpha}, u, v\rangle$ in the coordinate representation read ($\Delta = 0$ in equations (17) of the third paper of [11])

$$\langle \boldsymbol{q}|\boldsymbol{\alpha}; \Lambda\rangle = \pi^{N/4} \exp(\gamma + \tilde{\boldsymbol{\nu}}\boldsymbol{q} - \tfrac{1}{2}\boldsymbol{q}\tilde{\mu}\boldsymbol{q}) \tag{47}$$

where $\tilde{\mu}$ is $N \times N$ matrix, $\tilde{\mu} = \mathrm{i}\lambda_p^{-1}\lambda_q$, $\tilde{\boldsymbol{\nu}}$ is $N$ vector, $\tilde{\boldsymbol{\nu}} = (1/\sqrt{2})(\lambda_q^\dagger - \lambda_p^{-1}\lambda_q\lambda_p^\dagger)\boldsymbol{\alpha}$ and

$$\gamma = -\frac{1}{2}|\boldsymbol{\alpha}|^2 + \frac{\mathrm{i}}{4}\boldsymbol{\alpha}(\lambda_p^*\lambda_p^{-1}\lambda_q\lambda_p^\dagger - \lambda_q^*\lambda_p^\dagger)\boldsymbol{\alpha}.$$

At $\Lambda = 1$ the RIS (47) coincide with canonical CS $|\boldsymbol{\alpha}\rangle$ in the coordinate representation. The multimode states (47) in different parametrizations were also considered in several papers under the names multimode squeezed states [33] or multimode/polymode correlated states [19, 34, 35] or Gaussian pure states [35].

It is worth noting that for canonical RIS the condition (45) $\sim$ (19) is not only sufficient, but also necessary, i.e. all $\boldsymbol{Q}$-RIS are eigenstates of $a_j' = u_{jk}a_k + v_{jk}a_{jk}^\dagger$ for some $u_{jk}$ and $v_{jk}$. This can be proved by using the diagonalization of $\sigma(\boldsymbol{Q}, \rho)$. Indeed, let $\sigma' = \sigma(\boldsymbol{Q}', \rho) = \sigma(\boldsymbol{Q}, \rho')$ be diagonal. Here $\rho' = U(\Lambda)\rho U^\dagger(\Lambda)$, where $U$ is the methaplectic unitary operator. Since the mean commutator matrix $C$ is now constant, $\det C = 4^{-N}$ the equality in RUR is $\det\sigma = \det\sigma' = \prod_j^N \sigma_{jj}'\sigma_{N+j,N+j}' = 4^{-N}$. Since for every $j$ the product $\sigma_{jj}'\sigma_{N+j,N+j}'$ is greater than or equal to $\frac{1}{4}$ we obtain that all products should be equal to $\frac{1}{4}$. But this is only possible iff $\rho$ is a pure multimode CS for new variables $\boldsymbol{Q}'$, that is $\rho$ is pure state, methaplectically equivalent to multimode CS for old variables $\boldsymbol{Q}$, $\rho = U(\Lambda)|\boldsymbol{\alpha}\rangle\langle\boldsymbol{\alpha}|U^\dagger(\Lambda)$.

Consider briefly the uncertainty matrix of canonical observables $\sigma(\boldsymbol{Q}, \rho)$. Since $Q_\mu$ satisfy the requirements of proposition 2 the $\sigma(\boldsymbol{Q}, \rho)$ is positive definite. Therefore it can be diagonalized by means of linear canonical transformation in any state $\rho$ and it obeys inequalities (16). In $\boldsymbol{Q}$-RIS $|\boldsymbol{\alpha}, \Lambda\rangle$ the dispersion matrix $\sigma(\boldsymbol{Q}; \boldsymbol{\alpha}, \Lambda)$ has further properties. The main one is that $\sigma(\boldsymbol{Q}; \boldsymbol{\alpha}, \Lambda)$ is symplectic itself. Indeed we have

$$\sigma(\boldsymbol{Q}; \boldsymbol{\alpha}, \Lambda) = \sigma(\boldsymbol{Q}'; \boldsymbol{\alpha}, 1) = \Lambda\sigma(\boldsymbol{Q}; \boldsymbol{\alpha}, 1)\Lambda^T \tag{48}$$

where $\sigma(\boldsymbol{Q}; \boldsymbol{\alpha}, 1)$ is the uncertainty matrix in multimode canonical CS $|\boldsymbol{\alpha}\rangle$. The latter is evidently proportional to the unity, $\sigma(\boldsymbol{Q}; \boldsymbol{\alpha}, 1) = \frac{1}{2}$ and therefore if $\Lambda$ is symplectic then $2\sigma(\boldsymbol{Q}; \boldsymbol{\alpha}, \Lambda)$ is also symplectic. We express $\sigma$ in terms of $N \times N$ uncertainty matrices $\sigma_{pp}$, $\sigma_{qq}$, $\sigma_{qp}$ and $\sigma_{pq} = \sigma_{qp}^T$

$$\sigma(\boldsymbol{Q}) = \begin{pmatrix} \sigma_{pp} & \sigma_{pq} \\ \sigma_{qp} & \sigma_{qq} \end{pmatrix} \tag{49}$$

and write the symplectic properties of $\sigma(\boldsymbol{Q}; \boldsymbol{\alpha}, \Lambda)$ in $N \times N$ matrix form,

$$\sigma_{pp}\sigma_{qp} - \sigma_{pg}\sigma_{pp} = 0 = \sigma_{qp}\sigma_{qq} - \sigma_{qq}\sigma_{pq} \qquad \sigma_{pp}\sigma_{qq} - \sigma_{pq}^2 = \tfrac{1}{4}. \tag{50}$$

For $N = 1$ the last equality is just the equality in the Schrödinger relation (3), the first two being satisfied identically in any state.

For boson systems it is of interest to consider observables which are quadratic combinations of creation and annihilation operators $a_j^\dagger$ and $a_k$ (or equivalently of $p_j$ and $q_k$). Quadratic combinations

$$K_{jk} = \tfrac{1}{2}a_ja_k \qquad K_{jk}^\dagger = \tfrac{1}{2}a_k^\dagger a_j^\dagger \qquad K_{jk}^{(3)} = \tfrac{1}{4}(a_j^\dagger a_k + a_k^\dagger a_j) \tag{51}$$

close the simple noncompact algebra $sp(N, R)$ [6], the noncompact elements being spanned by lowering and raising operators $K_{jk}$ and $K_{jk}^\dagger$. In the one-mode case $sp(1, R) \sim su(1, 1)$ and

$$\tfrac{1}{2}a^2 = K_- \qquad \tfrac{1}{2}a^{\dagger 2} = K_+ \qquad \tfrac{1}{4}(2a^\dagger a + 1) = K_3. \tag{52}$$

Operators (51) are generators of the methapletic group $Mp(N, R)$, which covers the $Sp(N, R)$. $sp(N, R)$ RIS in representation (51) should be called the multimode squared amplitude RIS. RIS for the quadratures $X_{jk}$ and $Y_{jk}$ of $K_{jk}$, $K_{jk} = X_{jk} + \mathrm{i}Y_{jk}$ (shortly $K_{jk}$-RIS), are eigenstates of $N \times N$ complex combinations of lowering and raising operators $K_{jk}$ and $K^\dagger_{jk}$ and according to our general result they contain group-related $Mp(N, R)$ CS with maximal symmetry, $|\psi(g)\rangle = U(g)|0\rangle$, $U(g) \in Mp(N, R)$, the extremal vector being the multimode boson vacuum $|\mathbf{0}\rangle$ (these CS coincide with multimode squeezed vacuum states [19, 33, 34]). On the other hand, $Mp(N, R)$ CS are annihilated by all $a'_j = U(g)a_jU^{-1}(g) = u_{jk}a_k + v_{jk}a^\dagger_k$. Hereafter we obtain the property that $Mp(N, R)$ CS with maximal symmetry are simultaneously $h_N$ and $sp(N, R)$ RIS (i.e. amplitude and squared amplitude multimode RIS, double IS). In the coordinate representation and in the parametrization by $\lambda_q$ and $\lambda_p$ ($\boldsymbol{a}' = \lambda_q\boldsymbol{q} + \lambda_p\boldsymbol{p}$) these multimode double IS are given by formula (47) with $\boldsymbol{\alpha} = 0$.

Another explicit example of $sp(N, R)$ RIS is given by multimode squeezed Fock states $U(g)|\boldsymbol{n}\rangle$, where $U(g) \in Mp(N, R)$. Indeed, Fock states $|\boldsymbol{n}\rangle$ are eigenstates of Hermitian $Mp(N, R)$ generators $K^{(3)}_{jj} = a^\dagger_j a_j/2$ (see equation (51)), therefore $U(g)|\boldsymbol{n}\rangle$ are eigenstates of Hermitian operators $U(g)K^{(3)}_{jj}U(g)^\dagger$ which are real linear combinations of all $Mp(N, R)$ generators (follows from the BCH formula). From section 3 we know that this eigenvalue property is sufficient for the equalities $\det\sigma = \det C = 0$, i.e. the squeezed Fock states are $sp(N, R)$ RIS for all Hermitian quadratures of operators (51). Multimode squeezed Fock states $U(g)|\boldsymbol{n}\rangle$ were constructed in the last two papers of [11], where the $Mp(N, R)$ operator $U(g)$ was taken as the evolution operator $U(t)$ of the general $N$-dimensional quadratic quantum system (in coordinate representation the states $\langle\boldsymbol{q}|U(g)|\boldsymbol{n}\rangle$ were expressed as the product of $\langle\boldsymbol{q}|\mathbf{0}\rangle$ (see equation (47)) and a Hermite polynomial of $N$ variables). Note that squeezed Fock states are $sp(N, R)$ RIS and not $h_N$ RIS and squeezed Glauber CS are $h_N$ RIS and not $sp(N, R)$ RIS. Only squeezed vacuum states are simultaneously $sp(N, R)$ RIS and $h_N$ RIS ($h_N$ RIS = $\boldsymbol{Q}$-RIS).

Recently, attention was paid, in the physical literature, to multimode even and odd CS [36] $|\boldsymbol{\alpha}\rangle_\pm = N_\pm(|\boldsymbol{\alpha}\rangle \pm |-\boldsymbol{\alpha}\rangle)$, where $|\boldsymbol{\alpha}\rangle = D(\boldsymbol{\alpha})|\mathbf{0}\rangle$ is Glauber multimode CS. We readily see that these $|\boldsymbol{\alpha}\rangle_\pm$ are eigenstates of all $K_{jk}$, equation (51), and therefore are noncorrelated squared amplitude RIS with equal uncertainties of quadratures of $K_{jk}$. It is the set of all $sp(N, R)$ $K_{jk}$-RIS which is a natural extension of that of multimode even and odd CS $|\boldsymbol{\alpha}\rangle_\pm$, incorporating also the multimode squeezed vacuum states $|\mathbf{0}, u, v\rangle$ and Glauber CS $|\boldsymbol{\alpha}\rangle$. Unlike the even and odd CS $|\boldsymbol{\alpha}\rangle_\pm$, the $K_{jk}$-RIS (being eigenstates of combinations $u_{jk}a_ja_k + v_{jk}a^\dagger_ja^\dagger_k$) can exhibit strong squeezing in quadratures of $a_ja_k$ and therefore can be called *multimode squared amplitude squeezed states* in complete analogy to the well known case of multimode (amplitude) squeezed states [19, 33, 34].

We stress that the set of all $sp(N, R)$ RIS, and even the set of the $K_{jk}$-RIS is much larger than the set of $Mp(N, R)$ CS $U(g)|0\rangle$. The problem can be solved entirely in the one mode case, $N = 1$, using the Glauber CS representation, in which $a = \mathrm{d}/\mathrm{d}\alpha$, $a^\dagger = \alpha$ [30, 31]. The resulting even states take the form (37) with the replacements $k = \frac{1}{4}$ and $\eta = \alpha^2/2$, the normalizability conditions remaining the same as (39). Some particular sets of one-mode squared amplitude squeezed states are constructed and discussed in [17]. Generalized one-mode even and odd CS $|z, u, v; \pm\rangle$ were first constructed in the second paper of [19] as even and odd solutions of the eigenvalue equation $(ua^2 + va^{\dagger 2})|z, u, v; \pm\rangle = z|z, u, v; \pm\rangle$ with complex parameters $u$ and $v$. The eigenvalue problem for operators $(a + \zeta a^\dagger)^2$ ($\zeta \in \mathbb{C}$ was considered in [37].

The RIS, which are not group-related CS, exhibit many physical properties which group CS lack. One such property is squeezing in the fluctuation of group generators. Squeezing in the fluctuation of $X_\mu$ in a state $|\psi\rangle$ occurs if $|\psi\rangle$ is close (by norm form example) to an eigenstate of $X_\mu$ since the (squared) variance $\Delta^2 X_\mu = \sigma_{\mu\mu}$ of $X_\mu$ vanishes in eigenstates of $X_\mu$ *only* [15]. Therefore if in RIS which is eigenstate of $\beta_\nu X_\nu$ all but $\beta_\mu$ tend to 0 then $\Delta X_\mu$ should tend to 0. In group CS with symmetry it is not always possible to let all but $\beta_\mu$ to tend to 0 due to constrain (31) (it is trivially possible if $X_\mu$ itself has $|\psi_0\rangle$ as its eigenstate). In the case of $su(1,1)$ we have explicit solutions $|z,u,v,w;k\rangle$, equation (37) and CS $|\tau;k\rangle$ and one can verify the above statement: the variances of $K_{1,2}$ in CS are greater than $k$ for any $\tau$ [15], while for example for $k = \frac{1}{4}$ the $K_1$–$K_2$ IS $|z,u,v;\frac{1}{4}\rangle$ with $z = -1$, $u = \sqrt{1+x^2}$, $v = -x < 0$ exhibit strong squeezing in $K_2$ ($\Delta K_2$ is monotonically decreasing when $x$ increases). Moreover, one can find IS which exhibit $K_1$ ($K_2$) and $q$ ($p$) squeezing (joint amplitude and squared amplitude squeezing) simultaneously. Sub-Poissonian statistics also occurs in IS $|z,u,v,w;\frac{1}{4}\rangle$. In greater detail, nonclassical properties of $SU(1,1)$ IS (for $k = \frac{1}{4}, \frac{3}{4}$) are discussed (and illustrated by several graphics) in [31].

By means of four boson operators $a, b, a^\dagger, b^\dagger$ one can construct quadratic combinations which close $su(1,1)$ (the representations with Bargman index $k = (1 + |n_a - n_b|)/2 = 1/2, 1, \ldots$, considered in the previous subsection) or $su(2)$ algebra (the Schwinger realization), which are subalgebras of $sp(4,R)$, equation (51) for $N = 2$. Currently physical properties of various $su(1,1)$ and $su(2)$ SIS of two-mode boson/photon system are being discussed (see [38–40] and references therein). We note that the result of [40]: $K_2$–$K_3$ two-mode IS which are not $SU(1,1)$, group CS can improve the sensitivity in the interferometric measurements. Several schemes of generation of SIS for $su(1,1)$ or $su(2)$ operators in two-mode quadratic boson representations were considered recently [38–40]. But so far no scheme for generating $K_1$–$K_2$ one-mode SIS has been presented. It seems natural to generate these SIS from experimentally available Glauber CS or even and odd CS [8] acting on the latter by the squared amplitude squeeze operator $S(u,v)$, equation (27). For this purpose, however, one has to look for a nonunitary evolution process, since here the squeeze operator $S(u,v)$ is only isometric [31].

## 6. Concluding remarks

We have shown that the uncertainty matrix for $n$ observables $X_\mu$ can always be diagonalized by a linear transformation of $X_\mu$. For the case of spin-component operators this means that spin covariances are of a pure coordinate origin and correlated spin states are unitary equivalent to noncorrelated states. When the uncertainty matrix is positive definite (as is the case for example of the $q$-deformed multimode boson system with $q > 0$, in particular, the case of canonical boson system, $q = 1$) it can be diagonalized by means of symplectic transformations. Using the above diagonalization property a new family of uncertainty relations for positive definite uncertainty matrices is established.

The Robertson $n$-dimensional relation for the uncertainty matrix, equation (1), is shown to be efficient at generalizing the basic properties of Glauber coherent states (CS) to an arbitrary system of observables $X_\mu$. For an even number $n$ of observables this relation is minimized in a state $|\psi\rangle$ if $|\psi\rangle$ is an eigenstate of $n/2$ independent complex combinations of $X_\mu$. For any (even or odd) $n$ the minimization occurs in states which are eigenvectors of a real combination of $X_\mu$. When $X_\mu$ close a semisimple Lie algebra, the set of states which minimize the Robertson inequality (called here Robertson intelligent states (RIS)), contain the corresponding group-related CS with symmetry. CS with maximal symmetry are also contained in RIS for the quadratures of Weyl lowering and raising operators. Thus, it is the Robertson uncertainty relation that brings together the three ways of generalization of Glauber CS [3] to the level of $n$ observables.

RIS which are not group-related CS can exhibit interesting physical properties. One such universal property to be distinguished from CS is the strong squeezing of group generators. In this way the multimode squared amplitude squeezed states are naturally introduced as $sp(N, R)$ RIS. Squared amplitude RIS can exhibit both linear and quadratic squeezing as we have shown in the example of $K_1$–$K_2$ IS. Such joint squeezing of noncommuting observables could be useful in optical communications and interferometric measurements since the field in such squeezed states is better determined—this should be considered elsewhere. The problem of generating RIS for two $su(1, 1)$ and $su(2)$ observables was discussed in [38–40]. In this connection we note the principle possibility to generate for example $K_1$–$K_2$ squared amplitude IS by means of isometric (nonunitary) evolution operators.

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